

Equilibrium Cross Section of Returns

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We construct a dynamic general equilibrium production economy to explicitly link expected stock returns to firm characteristics such as firm size and the book-to-market ratio. Stock returns in the model are completely characterized by a conditional capital asset pricing model (CAPM). Size and book-to-market are correlated with the true conditional market beta and therefore appear to predict stock returns. The cross-sectional relations between firm characteristics and returns can subsist even after one controls for typical empirical estimates of beta. These findings suggest that the empirical success of size and book-to-market can be consistent with a single-factor conditional CAPM model.

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I. Introduction

The cross-sectional properties of stock returns have attracted considerable attention in recent empirical literature in financial economics. One of the best-known studies, by Fama and French (1992), uncovers the relations between firm characteristics such as book-to-market ratio and firm size and stock returns, which appear to be inconsistent with the standard capital asset pricing model (CAPM). Despite their empirical success, these simple statistical relations have proved very hard to rationalize, and their precise economic source remains a subject of debate.¹

We construct a dynamic stochastic general equilibrium one-factor model in which firms differ in characteristics such as size, book value, investment, and productivity, among others. It establishes an explicit economic relation between firm-level characteristics and stock returns. The simple structure of our model provides a parsimonious description of the firm-level returns and makes it a natural benchmark for interpreting many empirical regularities.

First, we show that our one-factor equilibrium model can still capture the ability of book-to-market and firm value to describe the cross section of stock returns. These relations can subsist after one controls for typical empirical estimates of conditional market beta. Second, we find that, in our model, the cross-sectional dispersion in individual stock returns is related to the aggregate stock market volatility and business cycle conditions. Third, we show that the size and book-to-market return premia are inherently conditional in their nature and likely countercyclical.

Our theoretical approach builds on the work of Berk et al. (1999), who construct a partial equilibrium model also based on the ideas of time-varying risks to explain cross-sectional variations of stock returns. However, our work differs along several important dimensions. First, ours is a single-factor model in which the conditional CAPM holds, whereas the model of Berk et al. introduces a second risk factor in addition to the market portfolio. The simple structure of our model allows us to derive an explicit link between the beta (and hence returns) and firm characteristics such as size and book-to-market. Instead of appealing to multiple sources of risk, we emphasize the role of beta mismeasurement in generating the observed cross-sectional relations between the Fama and French factors and stock returns. Second, by

¹ Campbell, Lo, and MacKinlay (1997), Cochrane (1999), and Campbell (2000) review the related literature. Various competing interpretations of observed empirical regularities include, among others, Lo and MacKinlay (1988), Fama and French (1993, 1995, 1996), Lakonishok, Shleifer, and Vishny (1994), Berk (1995), Kothari, Shanken, and Sloan (1995), MacKinlay (1995), Jagannathan and Wang (1996), Berk, Green, and Naik (1999), Liew and Vassalou (2000), and Lettau and Ludvigson (2001).

explicitly modeling the production and investment decisions of the firms, we are able to integrate our cross-sectional analysis into a general equilibrium model that allows us to present a self-consistent account of the business cycle properties of returns.

Our work belongs to a growing literature that explores the implications of production and investment on the cross section of returns. In addition to Berk et al. (1999), recent examples include Cochrane (1996), Gomes, Yaron, and Zhang (2002), and Zhang (2002). More broadly, this paper is also related to a variety of recent papers that focus on the asset pricing implications of production and investment in the time series. Examples of this line of research include Bossaerts and Green (1989), Cochrane (1991), Naik (1994), Rouwenhorst (1995), Coleman (1997), Jermann (1998), and Kogan (2000, 2001). To the best of our knowledge, however, this is the first work aiming directly at explaining the cross-sectional variations of stock returns from a structural general equilibrium perspective.

II. The Model

We develop a general equilibrium model with heterogeneous firms. There are two types of agents: a single representative household and a large number of competitive firms producing a single consumption good.

A. Production Sector

Production of the consumption good takes place in basic productive units, which we label *projects*. New projects are continuously arriving in the economy. Projects are owned by firms, and each firm operates a number of individual projects of different characteristics.

Existing Projects

Let \mathcal{I}_t denote the set of all projects existing at time t , and let i be the index of an individual project. Projects expire randomly according to an idiosyncratic Poisson process with common hazard rate δ (we define the arrival of new projects below). Existing projects have two individual features: productivity and scale.

Productivity is driven by a component common to all projects, x_t , and a project-specific element, ϵ_{it} . We assume that x_t follows the linear mean-reverting process,

$$dx_t = -\theta_x(x_t - \bar{x})dt + \sigma_x dB_{x,t} \quad (1)$$

and that ϵ_{it} is driven by a square root process,

$$d\epsilon_{it} = \theta_\epsilon(1 - \epsilon_{it})dt + \sigma_\epsilon\sqrt{\epsilon_{it}}dB_{it}, \quad (2)$$

where B_{xt} and B_{it} are standard Brownian motions.² We assume that the idiosyncratic productivity shocks are independent of the economywide productivity shock, that is, $dB_{xt}dB_{it} = 0$ for all i . We make a further assumption that if projects i and j are owned by the same firm (see below), $dB_{it}dB_{jt} = dt$; otherwise we set $dB_{it}dB_{jt} = 0$.

While the specific nature of processes (1) and (2) is merely convenient, mean reversion is important. At the aggregate level, it is necessary to ensure that the growth rate of output does not explode, a result consistent with standard findings in the growth literature (Kaldor 1961). At the firm level, mean reversion is required to obtain a stationary distribution of firms in equilibrium and is consistent with the evidence suggesting that growth rates decline with size and age (Evans 1987; Hall 1987).

The scale of a project, denoted k_p , is set at the time of creation, and it remains fixed throughout the life of the project. Given its scale and productivity, each project generates a flow of output (cash flows) at rate $\exp(x_t)\epsilon_{it}k_i$. We compute the net present value of the future stream of cash flows associated with the project, $P(x_p, \epsilon_{ip}, k_i)$. Let $M_{t,t+s}$ denote the pricing kernel, which determines prices of all financial assets. If an asset pays a flow of dividends at rate Z_s , its time t price is given by

$$E_t \left[\int_0^\infty M_{t,t+s} Z_{t+s} ds \right].$$

PROPOSITION 1. Project valuation.—The value of an existing project i is given by

$$\begin{aligned} P(x_p, \epsilon_{ip}, k_i) &= E_t \left[\int_0^\infty e^{-\delta s} M_{t,t+s} (e^{x_{t+s}} \epsilon_{i,t+s} k_i) ds \right] \\ &= k_i [p(x_t) + \tilde{p}(x_t)(\epsilon_{it} - 1)], \end{aligned} \quad (3)$$

where $p(x_t)$ and $\tilde{p}(x_t)$ are defined as

$$p(x_t) = E_t \left[\int_0^\infty e^{-\delta s} M_{t,t+s} e^{x_{t+s}} ds \right] \quad (4)$$

² The process in (1) is chosen to possess a stationary long-run distribution with constant instantaneous volatility. The advantage of (2) is that the conditional expectation of ϵ_{it} is an exponential function of time and a linear function of the initial value ϵ_{i0} , which facilitates computation of individual stock prices below. An additional advantage of this process is that its unconditional mean is independent of θ_ϵ and σ_ϵ , which simplifies the calibration.

and

$$\tilde{p}(x_t) = E_t \left[\int_0^{\infty} e^{-(\delta+\theta_\epsilon)s} M_{t,t+s} e^{x_{t+s}} ds \right]. \quad (5)$$

Proof. See Appendix A.

In (3), $e^{x_{t+s}} \epsilon_{i,t+s} k_i$ is the cash flow rate of project i , which is valued using the pricing kernel $M_{t,t+s}$. The factor $e^{-\delta s}$ captures the fact that existing projects expire randomly at rate δ . The present value $p(x)$ represents the component of the value of an existing project attributable to the level of aggregate productivity, and $\tilde{p}(x)$ captures the sensitivity of the value of the project to the idiosyncratic component of its productivity. Note that $\tilde{p}(x)$ and $p(x)$ differ only in the rate of discount, which implies that $\tilde{p}(x) < p(x)$, for all x . In addition, as $\theta_\epsilon \rightarrow 0$, we have that $\tilde{p}(x) \rightarrow p(x)$ and $P(x, \epsilon_{it}, k_i) = p(x) \epsilon_{it} k_i$.

New Projects

At the aggregate level, new potential projects arrive continuously. These projects can be adopted at time t with an investment cost of $e_{it} k_p$, where e_{it} is the unit cost of adoption. If the project is not adopted, it disappears.

We assume that during any period $[t, t + dt]$, multiple projects arrive with various values of their unit cost e_{it} . For simplicity, we are assuming that the arrival rate of new projects is independent of project unit cost. The production scale of all new projects with unit cost between e and $e + de$ arriving during the time interval $[t, t + dt]$ adds up to $h_t de dt$, where h_t determines the instantaneous arrival rate of new projects.

We make two additional simplifying assumptions regarding the scale and productivity of these new projects. First, all projects of the same vintage have the same scale, k_p . This scale is chosen to ensure that the number of projects per firm has a stationary distribution (see App. B for details). Second, the initial productivity of a new project is drawn from the long-run distribution implied by (2), but only *after* the project is adopted. Given these assumptions, the value of a new project at time t immediately *before* the project is adopted is given by

$$E[P(x, \epsilon_{it}, k_i) | x_t] = k_i p(x_t)$$

since $E[\epsilon_{it} | x_t] = 1$.

Firms

Projects are owned by infinitely lived firms. We assume that the set of firms \mathcal{F} is exogenously fixed and let f be the index of an individual firm.

Each firm owns a finite set of individual projects, \mathcal{I}_t , which changes over time as new projects get adopted and existing projects expire.

Firms are financed entirely by equity, and outstanding equity of each firm is normalized to one share. We denote the firm's f stock price at time t by V_t . Stocks represent claims on the dividends paid by firms to shareholders, and we assume that the dividend equals the firm's output net of investment costs. We assume that firms are competitive and their objective is to maximize the market value of their equity.

Regardless of its unit cost, each new project is allocated to a randomly chosen firm. Hence, all firms have an equal probability of receiving a new project at any point in time. Assuming that all firms are equally likely to receive new projects allows for tractability, but it is not crucial. Qualitatively, we need firm growth to be negatively related to size, a fact well documented in the data.

While firms do not control the scale or productivity of their projects, they make investment decisions by selecting which of these new projects to adopt. If the firm decides to invest in a new project, it must incur the required investment cost, which in turn entitles it to the permanent ownership of the project. These investment decisions are irreversible, and investment cost cannot be recovered at a later date.

For the firm, the arrival rate of new projects is independent of its own past investment decisions. Thus the decision to accept or reject a specific project has no effect on the individual firm's future investment opportunities and therefore can be made using a standard net present value rule. Given that the present value of future cash flows from a new project at time t equals $k_t p(x_t)$, it follows that new projects are adopted if and only if their unit investment cost is below $p(x_t)$:

$$e_{it} \leq p(x_t). \quad (6)$$

Hence, the decision to adopt new projects can be summarized by a function of aggregate productivity, x_t . Figure 1 illustrates this.

The value of the firm, V_t , can be viewed as the sum of the present value of output from existing projects, V_t^a , plus the present value of dividends (output net of investment) from future projects, V_t^o . With the terminology from Berk et al. (1999), V_t^a represents the value of *assets in place*, defined as

$$V_t^a = \sum_{i \in \mathcal{I}_t} P(x_t, \epsilon_{it}, k_i) = \sum_{i \in \mathcal{I}_t} k_i [p(x_t) + \tilde{p}(x_t)(\epsilon_{it} - 1)], \quad (7)$$

whereas $V_t^o = V_t - V_t^a$ can be interpreted as the value of *growth options*.

For future use we also define the book value of a firm as the sum of book values of the firm's (active) individual projects, $B_t = \sum_{i \in \mathcal{I}_t} e_i k_i$, and the book value of a project is defined as the associated investment cost $e_i k_i$.

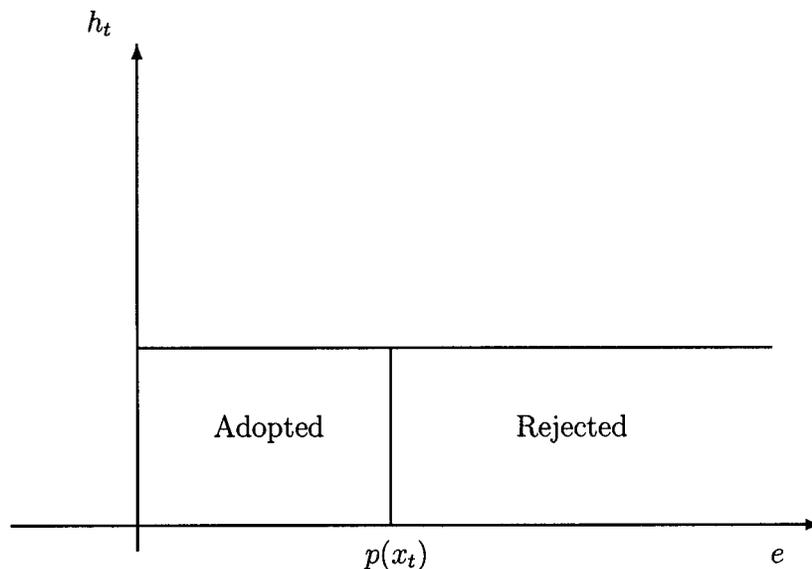


FIG. 1.—Arrival of new projects. This figure illustrates the project arrival rate h_t as a function of its unit cost e . The function $p(x_t)$ denotes the component of the value of an existing project attributable to the level of aggregate productivity.

Aggregation

Let $\int_{q_t} \cdot di$ denote the aggregation operator over projects, and define the aggregate scale of production in the economy, K_t , as

$$K_t \equiv \int_{q_t} k_t di.$$

It follows that aggregate output, Y_t , is given by

$$Y_t = \int_{q_t} \exp(x_t) k_t \epsilon_{it} di = \exp(x_t) K_t \quad (8)$$

where the second equality follows from the fact that the project scale, k_t , is fixed at time of creation and is independent of idiosyncratic productivity, ϵ_{it} , and the law of large numbers applied to ϵ_{it} 's, which are independently and identically distributed with unit mean.³ Equation (8)

³ Feldman and Gilles (1985) formalize the law of large numbers in economies with countably infinite numbers of agents by aggregating with respect to a finitely additive measure over the set of agents. Judd (1985) demonstrates that a measure and the corresponding law of large numbers can be meaningfully introduced for economies with a continuum of agents.

is then consistent with our interpretation of x_t as the aggregate productivity shock.

Since active projects expire at rate δ , whereas new projects are adopted only if their creation cost is below $p(x_t)$, the total scale of projects in the economy evolves according to

$$dK_t = -\delta K_t dt + \left[\int_0^{p(x_t)} h_t de \right] dt. \quad (9)$$

Balanced growth requires that the aggregate arrival rate, h_t , be proportional to the aggregate scale of existing projects, K_t . Formally, we assume that $h_t = zK_t$, where the parameter z governs the quality of the investment opportunity set.

Given our assumptions about h_t , (9) implies that the change in the total scale of production is given by

$$dK_t = -\delta K_t dt + zK_t p(x_t) dt \quad (10)$$

and the amount of resources used in the creation of new projects, I_t , equals

$$I_t = I(x_t) \equiv \int_0^{p(x_t)} ezK_t de = \frac{1}{2} zK_t [p(x_t)]^2. \quad (11)$$

The aggregate dividend of firms equals the aggregate output net of aggregate investment and is given by

$$D_t \equiv Y_t - I_t = \{e^{x_t} - \frac{1}{2} z [p(x_t)]^2\} K_t. \quad (12)$$

Note that since $p(x)$ is increasing in x , this implies that more expensive projects are adopted only in good times, when x_t is high. This rising cost of investment is then similar to the result obtained in a standard convex adjustment cost model. Together, our assumptions about productivity and costs guarantee that individual investment decisions can be aggregated into a linear stochastic growth model with adjustment costs. This provides a tractable setting for addressing the behavior of the cross section of returns.

The production environment in our model differs from that in Berk et al. (1999) in a number of critical aspects. First, Berk et al. simply assume that cash flows of existing projects are independently distributed over time and have a constant beta with respect to an exogenous stochastic pricing kernel that is driven by serially independent shocks. Second, in their model, exogenous fluctuations in real interest rates are driven by a separate first-order stationary Markov process, creating an additional source of risk. As a consequence, the value of existing assets is exposed to two risk factors: while the capital gains component of

returns is related to fluctuations in interest rates, cash flows from existing projects covary with the shocks to the pricing kernel. Moreover, the present value of future projects, that is, the value of growth options, depends only on the current level of the interest rate, since unexpected changes in the pricing kernel are independently and identically distributed. Thus, while the value of existing assets is exposed to both sources of risk, the value of growth options has a positive loading only on the level of the interest rate.

B. Households

The economy is populated by identical competitive households, which derive utility from the consumption flow of the single good, C_t . The entire population can then be modeled as a single representative household, and we assume that this household has standard time-separable iso-elastic preferences,

$$E_0 \left[\frac{1}{1-\gamma} \int_0^{\infty} e^{-\lambda t} C_t^{1-\gamma} dt \right], \quad (13)$$

where λ is the subjective rate of discount and γ is the coefficient of relative risk aversion. Households do not work and derive income from accumulated wealth, W_t . We assume that there exists a complete set of financial markets and there are no frictions and no constraints on short sales or borrowing. The term $M_{t,t+s}$ is the unique equilibrium pricing kernel, which determines prices of all financial assets.

The representative household maximizes the expected utility of consumption (13), taking the prices of financial assets as given. In a complete financial market, the budget constraint is given by

$$E_t \left[\int_0^{\infty} M_{t,t+s} C_{t+s} ds \right] \leq W_t. \quad (14)$$

Optimality conditions imply a well-known relation between the consumption policy and the pricing kernel:

$$M_{t,t+s} = e^{-\lambda s} \left(\frac{C_t}{C_{t+s}} \right)^{\gamma}. \quad (15)$$

C. The Competitive Equilibrium

With the description of the economic environment complete, we are now in a position to state the definition of the competitive equilibrium.

DEFINITION 1. *Competitive equilibrium.*—A competitive equilibrium is

summarized by the pricing kernel $M_{t,t+s}$, the optimal household consumption policy C_p , and firm investment policy, described by $p(x_t)$, such that the following conditions hold: (a) Optimization: (i) With the equilibrium asset prices taken as given, households maximize their expected utility (13), subject to the budget constraint (14). (ii) With the equilibrium asset prices taken as given, firms select new projects according to (6) and (4). (b) Market clearing: Representative household consumption equals the aggregate dividend, given by (12):

$$C_t = D_t \quad (16)$$

The competitive equilibrium has a very convenient structure. Since the cross-sectional distribution of firms has no impact on aggregate quantities, we characterize the optimal consumption and investment policies first and use them to compute the aggregate stock market value. Given the aggregate quantities, we then express explicitly the individual firm prices and returns.

Proposition 2 establishes that the optimal policies for consumption and investment can be characterized by a system of one differential equation and one algebraic equation.

PROPOSITION 2. *Equilibrium allocations.*—The competitive equilibrium is characterized by the optimal investment policy, described by $p(x)$ in (6), and consumption policy, $C(x, K)$, which satisfy

$$C(x, K) = \{e^x - \frac{1}{2}z[p(x)]^2\}K \quad (17)$$

and

$$p(x) = \{e^x - \frac{1}{2}z[p(x)]^2\}^\gamma \phi(x), \quad (18)$$

where the function $\phi(x)$ satisfies

$$e^x \{e^x - \frac{1}{2}z[p(x)]^2\}^{-\gamma} = [\lambda + (1 - \gamma)\delta + \gamma z p(x)]\phi(x) - \mathcal{A}[\phi(x)] \quad (19)$$

and $\mathcal{A}[\cdot]$ is the infinitesimal generator of the diffusion process x_t :

$$\mathcal{A}[g(x)] \equiv -\theta_x(x - \bar{x})g'(x) + \frac{1}{2}\sigma_x^2 g''(x). \quad (20)$$

Proof. See Appendix A.

This concept of general equilibrium is also one of the key novelties in our analysis relative to that of Berk et al. (1999), who instead proceed by keeping the pricing kernel, $M_{t,t+s}$, entirely exogenous, thus separating the optimal investment decisions from the consumption allocation.

D. Asset Prices

With the optimal allocations computed, we now characterize the asset prices in the economy, including the risk-free interest rate and both the aggregate and firm-level stock prices.

Aggregate Prices

The following proposition summarizes the results for the equilibrium values of the risk-free rate, r_t , and the aggregate stock market value, V_t .

PROPOSITION 3. *Equilibrium asset prices.*—The instantaneous risk-free interest rate is determined by

$$r_t = -\frac{E_t[M_{t,t+dt} - 1]}{dt} = \lambda + \gamma[zp(x_t) - \delta] + \gamma \frac{\mathcal{A}[C(x_t, K_t)]}{C_t} - \frac{1}{2}\gamma(\gamma + 1)\sigma_x^2 \left[\frac{\partial \ln C(x_t, K_t)}{\partial x_t} \right]^2. \quad (21)$$

The aggregate stock market value, V_t , can be computed as

$$\begin{aligned} V_t &= E_t \left[\int_0^\infty M_{t,t+s} D_{t+s} ds \right] \\ &= E_t \left[\int_0^\infty e^{-\lambda s} \left(\frac{C_t}{C_{t+s}} \right)^\gamma C_{t+s} ds \right] \\ &= \{e^{x_t} - \frac{1}{2}z[p(x_t)]^2\}^\gamma \psi(x_t) K_t, \end{aligned} \quad (22)$$

where the function $\psi(x)$ satisfies the differential equation

$$\lambda\psi(x) = \{e^x - \frac{1}{2}z[p(x)]^2\}^{1-\gamma} + (1 - \gamma)[zp(x) - \delta]\psi(x) + \mathcal{A}[\psi(x)],$$

and $\mathcal{A}[\cdot]$ is defined as in (20).

Proof. See Appendix A.

While these exact conditions are somewhat technical, the intuition behind them is quite simple. The instantaneous risk-free interest rate is completely determined by the equilibrium consumption process of the representative household and its implied properties for the pricing kernel. The aggregate stock market value represents a claim on the future stream of aggregate dividends, D_t , paid out by firms, which in equilibrium must equal aggregate consumption, C_t .

Finally, given (22), we can also define the process for cumulative aggregate stock returns as

$$\frac{dR_t}{R_t} = \frac{dV_t + D_t dt}{V_t}. \quad (23)$$

In addition to the definition above, the value of the stock market can also be viewed as a sum of two components. The first is the value of assets in place: the present value of output from existing projects. It is given by the expression

$$\begin{aligned} V_t^a &= \int_{\mathcal{F}} V_{jt}^a df \\ &= \int_{\mathcal{F}} \left\{ \sum_{i \in \mathcal{I}_{jt}} k_i [p(x_t) + \tilde{p}(x_t)(\epsilon_{it} - 1)] \right\} df \\ &= p(x_t) \int_{\mathcal{I}_t} k_i di + \tilde{p}(x_t) \int_{\mathcal{I}_t} k_i (\epsilon_{it} - 1) di \\ &= p(x_t) K_t \end{aligned} \quad (24)$$

where the last equality follows from applying the law of large numbers to ϵ_{it} . The difference between the aggregate market value and the value of assets in place is the value of aggregate growth options, defined as the present value of dividends from all projects to be adopted in the future. By definition, the value of aggregate growth options equals

$$V_t^o = V_t - V_t^a. \quad (25)$$

Firm-Level Stock Prices

Valuation of individual stocks is straightforward once the aggregate market value is computed. First, the value of a firm's stock is the sum of the value of assets in place for the firm, (7), and the value of growth options. Given our assumption that new projects are distributed randomly across all firms with equal probabilities, all firms will derive the same value from growth options. Hence, the value of growth options for each firm, V_{jt}^o , equals

$$V_{jt}^o = \bar{V}_t^o = \frac{1}{\int_{\mathcal{F}} 1 df} V_t^o. \quad (26)$$

We obtain the total value of the firm, V_{jt} , as

$$V_{jt} = \sum_{i \in \mathcal{I}_{jt}} k_i [\tilde{p}(x_i)(\epsilon_{it} - 1) + p(x_i)] + \bar{V}_t^o. \quad (27)$$

By relating individual firm value to market aggregates, the decomposition (27) implies that the instantaneous market betas of individual stock returns can also be expressed as a weighted average of market betas of three *economywide* variables, p , \tilde{p} , and V_t^o . Proposition 4 formally establishes this property.

PROPOSITION 4. *Market betas of individual stocks.*—Firm market betas are described by

$$\beta_{jt} = \tilde{\beta}_t^a + \frac{\bar{V}_t^o}{V_{jt}} (\beta_t^o - \tilde{\beta}_t^a) + \frac{1}{p(x_i)} \frac{K_{jt}}{V_{jt}} (\beta_t^a - \tilde{\beta}_t^a), \quad (28)$$

where

$$K_{jt} = \sum_{i \in \mathcal{I}_{jt}} k_i$$

and

$$\beta_t^a = \frac{\partial \log p_t / \partial x}{\partial \log V_t / \partial x}, \quad \tilde{\beta}_t^a = \frac{\partial \log \tilde{p}_t / \partial x}{\partial \log V_t / \partial x}, \quad \beta_t^o = \frac{\partial \log V_t^o / \partial x}{\partial \log V_t / \partial x}. \quad (29)$$

Proof. Since the market beta of a portfolio of assets is a value-weighted average of betas of its individual components, the expression for the value of the firm (27) implies that

$$\begin{aligned} \beta_{jt} &= \left(1 - \frac{V_{jt}^o}{V_{jt}}\right) \beta_{jt}^a + \frac{V_{jt}^o}{V_{jt}} \beta_t^o \\ &= \left(1 - \frac{V_{jt}^o}{V_{jt}}\right) [(1 - \pi_{jt}) \tilde{\beta}_t^a + \pi_{jt} \beta_t^a] + \frac{V_{jt}^o}{V_{jt}} \beta_t^o, \end{aligned}$$

where

$$\pi_{jt} = \frac{K_{jt}}{V_{jt}^a} \left(\frac{K_{jt}}{V_{jt}^a}\right)^{-1} = \frac{K_{jt}}{V_{jt}^a} \frac{1}{p(x_i)}.$$

Simple manipulation then yields (28). Q.E.D.

Stock Returns and Firm Characteristics

By definition, β^a is the market risk of aggregate assets in place, (24), and β^o is the market risk of aggregate growth options, (25). Potential future projects are valued as growth options because they have a positive net present value; that is, new potential projects are adopted only if

$p(x) - e > 0$. Since the volatility of $p(x) - e$ exceeds the volatility of $p(x)$, this leverage effect will likely imply that $\beta^o > \beta^a$. In this respect, our model differs from that of Berk et al. (1999) in that the risk of growth options in their model is relatively low, being entirely determined by the exogenous process for the interest rate.⁴

At the level of individual projects, according to the interpretation of the present values $p(x_i)$ and $\tilde{p}(x_i)$, β^a describes the component of systematic risk that is common to all existing projects, and β^o captures the cross-sectional differences between projects due to the idiosyncratic component of their productivity. The relation between these two aggregates is less immediate. By definition, $p(x_i)$ and $\tilde{p}(x_i)$ differ only with respect to the discount rate in the present value relations (4) and (5). Since the “effective duration” of the cash flows defining $p(x_i)$ exceeds that of $\tilde{p}(x_i)$, the relation between β^a and $\tilde{\beta}^a$ depends on the equilibrium term premium; specifically, a positive term premium will tend to raise β^a relative to $\tilde{\beta}^a$. In the calibrated version of our model, β^a actually exceeds $\tilde{\beta}^a$, as shown in figure 2. This implies that more productive projects, that is, those with higher values of ϵ_{it} , have lower systematic risk in our model.

Proposition 4 shows that the weights on the “aggregate” betas, β_t^a , $\tilde{\beta}_t^a$, and β_t^o , depend on the economywide variables $p(x_t)$ and V_t^o and, more important, on firm-specific characteristics such as the size, or value, of the firm, V_{jt} and the ratio of the firm’s production scale to its market value, K_{jt}/V_{jt} .

The second term in (28) creates an inverse relation between size and beta, as the weight on the beta of growth options, β_t^o , depends on the value of the firm’s growth options relative to its total market value. Firms with a small production scale, K_{jt} , derive most of their value from growth options, and their betas are close to β_t^o . Since all firms in our economy have identical growth options, the cross-sectional dispersion of betas due to the loading on β_t^o is captured entirely by the size variable V_{jt} . Large firms, on the other hand, derive a larger proportion of their value from assets in place; therefore, their betas are close to a weighted average of β_t^a and $\tilde{\beta}_t^a$. While this “size effect” is a result of our assumption about the distribution of growth options across firms, the effect will survive as long as V_{jt}^o/V_{jt} differs across firms, which requires only that growth options are less than proportional to size. Given the observed negative relation between firm size and growth (Evans 1987; Hall 1987), this seems quite plausible.

The last term in (28) shows that a part of the cross-sectional dispersion of market betas is also related to the firm-specific ratio of the scale of production to the market value, K_{jt}/V_{jt} , to a certain extent similar to

⁴ See Berk et al. (1999, n. 7) for a detailed discussion of this issue.

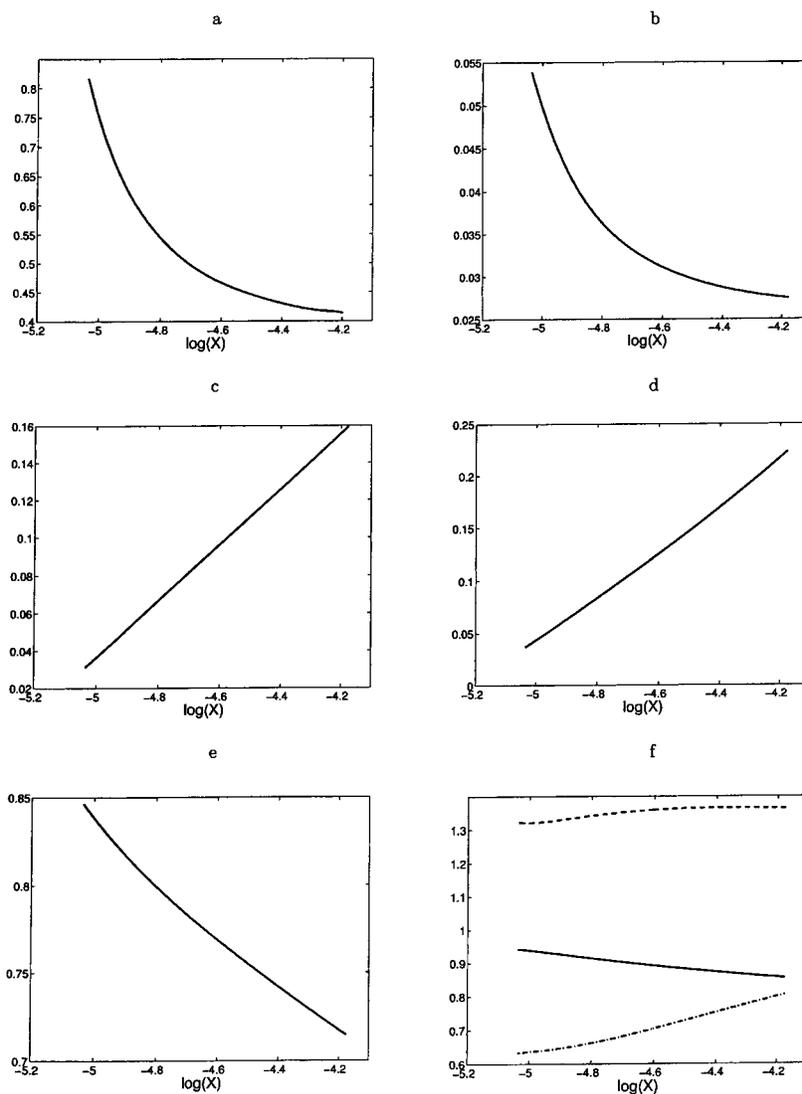


FIG. 2.—Some key aggregate variables in competitive equilibrium. This figure plots some key aggregate variables in competitive equilibrium. *a*, Market Sharpe ratio. *b*, Volatility of consumption growth. *c*, The function $p(x)$ or, equivalently, V^o/K in (24). *d*, The ratio of total market value to aggregate capital stock, V/K . *e*, The ratio of aggregate value of assets in place to total market value, V^o/V . *f*, Three aggregate-level betas: β^a (solid line), β^o (dashed-dotted line), and β^o (dashed line), defined in (29).

the empirical measure of the firm's book-to-market ratio.⁵ To see the intuition behind this result, consider two firms, 1 and 2, with the same market value. Since the value of growth options of these two firms is also identical, differences in their market risk are due only to the distribution of cash flows from the firms' existing projects. For simplicity, assume that each firm has a single active project. Let firm 1's project have a larger scale, so that firm 1 has a higher ratio K_{jt}/V_{jt} . Because the market value of a project is increasing in its idiosyncratic productivity, firm 1's project must have lower productivity than firm 2's project. As we have discussed above, more productive projects in our model have lower systematic risk; hence firm 2 should have a lower market beta than firm 1. This argument shows that in our model the book-to-market ratio measures the systematic risk of a firm's returns because of its relation to the productivity and systematic risk of a firm's existing projects.

In the argument above, we considered single-project firms. We show below that, more generally, the book-to-market ratio in our model is negatively related to firm profitability, defined as the ratio of a firm's output to its book value. See Section IVC below for more discussion on this relation.

Although the book-to-market ratio is commonly interpreted as an empirical proxy for the firm's growth options, the preceding discussion shows that this need not be the case. In our model, size serves as a measure of a firm's growth options relative to its total market value, and the book-to-market ratio captures the risk of the firm's assets in place. Berk et al. (1999) point out a similar effect in the context of their model, even though the structure of their economy is substantially different from ours.

Because of the single-factor nature of our model, the cross-sectional distribution of expected returns is determined entirely by the distribution of market betas, since returns on the aggregate stock market are instantaneously perfectly correlated with the consumption process of the representative household (and hence the pricing kernel; e.g., Breeden [1979]). Thus, if conditional market betas were measured with perfect precision, no other variable would contain additional information about the cross section of returns. However, equation (28) implies that if for any reason market betas were mismeasured (e.g., because the market portfolio is not correctly specified), then firm-specific variables such as firm size and book-to-market ratios could appear to predict the cross-sectional distribution of expected stock returns simply because they are related to the true conditional betas. In Section IV we generate

⁵ The ratio K_{jt}/V_{jt} can also be approximated by other accounting variables, e.g., by the earnings-to-price ratio.

an example within our artificial economy of how mismeasurement of betas can lead to a significant role of firm characteristics as predictors of returns.

III. Aggregate Stock Returns

In this section we evaluate our model's ability to reproduce a few key features of aggregate data on returns. While this is not our main objective, it seems appropriate to ensure that the model's implications for the time series of stock returns are reasonable before examining its cross-sectional properties. To guarantee this, we restrict the values of the seven aggregate-level parameters, γ , λ , δ , \bar{x} , θ_x , σ_x , and z , to approximately match seven key unconditional moments: the first two unconditional moments of stock returns, the risk-free rate and aggregate consumption growth, and the average level of the investment-to-output ratio. We then examine the implications of these choices for a number of conditional moments of asset returns.

A. Unconditional Moments

The values of the model parameters used in the simulation are as follows: the risk aversion coefficient, γ , 15; the time preference parameter, λ , 0.01; the rate of project expiration, δ , 0.04; the long-run mean of the aggregate productivity variable, \bar{x} , $\log(0.01)$; the quality of investment opportunities, z , 0.50; the volatility of the productivity variable, σ_x , 0.08; the rate of mean reversion of the productivity variable, θ_x , 0.275; the rate of mean reversion of the idiosyncratic productivity component, θ_e , 0.50; and the volatility of the idiosyncratic productivity component, σ_e , 2.00. Table 1 compares the implied moments of the key aggregate variables in the model with corresponding empirical estimates. We report both population moments, estimated by simulating a 300,000-month time series, and sample moments based on 200 simulations each with 70 years of monthly data. For the sample moments, in addition to point estimates and standard errors, we also report 95 percent confidence intervals based on empirical distribution functions from 200 simulations.

Essentially, our model captures the historical level and the volatility of the equity premium, while maintaining plausible values for the first two moments of the risk-free rate. Given the simple time-separable constant relative risk aversion utility and volatility in consumption growth of about 3 percent, this is possible only with a sizable degree of risk aversion (15). Since instantaneous consumption growth and aggregate stock returns are perfectly correlated in our single-factor model, this implies that the instantaneous Sharpe ratio of returns is approximately

TABLE 1
MOMENTS OF KEY AGGREGATE VARIABLES

	DATA		POPULATION		SAMPLE	
	Mean (1)	Standard Deviation (2)	Mean (3)	Standard Deviation (4)	Mean (5)	Standard Deviation (6)
$(C_{t+1}/C_t) - 1$	1.72	3.28	.85	3.22	.84 (.28) [.22 1.33]	3.06 (.26) [2.56 3.50]
r_t	1.80	3.00	1.30	4.33	1.34 (1.30) [-.63 4.23]	3.98 (.85) [2.55 5.73]
$\log R_t - \log r_t$	6.00	18.0	6.00	14.34	5.89 (1.32) [2.97 8.13]	15.28 (1.73) [11.80 18.58]
I_t/Y_t	.19		.23		.23 (.02) [.19 .26]	

NOTE.—This table reports unconditional means and standard deviations of consumption growth $([C_{t+1}/C_t] - 1)$, real interest rate (r_t), equity premium ($\log R_t - \log r_t$), and the mean of the investment-to-output ratio (I_t/Y_t). The numbers reported in cols. 1 and 2 are taken from Campbell, Lo, and MacKinlay (1997), except for the mean of the investment-to-output ratio, which equals the postwar average for the U.S. economy. The numbers reported in cols. 3 and 4 are population moments. These statistics are computed on the basis of 300,000 months of simulated data. Cols. 5 and 6 report the finite-sample properties of the corresponding statistics. We simulate 70-year-long monthly data sets, a length comparable to the sample length typically used in empirical research. Simulation is repeated 200 times, and the relevant statistics are computed for every simulation. Then we report the averages across the 200 replications. The numbers in parentheses are standard deviations across these 200 simulations, and the two numbers in brackets are 2.5th and 97.5th percentiles of the resulting empirical distribution, respectively. All numbers except those in the last row are in percentages.

equal to $15 \times 0.03 = 0.45$, which is close to its historical average. Figure 2 shows that the Sharpe ratio of the aggregate stock market returns in our model is also countercyclical, which is consistent with empirical facts.

Given our focus on the cross-sectional properties of returns, this seems to be an acceptable approximation. Despite the apparent success, however, it is unlikely that the model provides a precise account of the exact mechanism behind the empirical properties of the aggregate stock returns. Although our model generates a plausible level of stock market volatility, the time-separable nature of preferences implies that most of this variation is due to changes in the risk-free rate.⁶ In addition, the joint determination of consumption and output in our production-based asset pricing model implies a negative autocorrelation for consumption growth (rising from -0.02 after one quarter to -0.14 after four years), somewhat at odds with its (near) random walk pattern observed in the data.

⁶ Campbell (2003) discusses alternative preference specifications that can overcome this difficulty.

B. Conditional Moments

Proposition 4 shows that the cross-sectional distribution of firm betas is determined by a number of aggregate variables. To the extent that betas are linked to returns, this also implies a link between firm characteristics and stock returns. We now investigate our model's implications for these two relations.

Theoretically, our model implies perfect correlation between instantaneous stock returns and the pricing kernel. As a result, the aggregate market portfolio is instantaneously mean-variance efficient, and asset returns are characterized by a conditional CAPM. Quantitatively, we find that the unconditional correlation between the pricing kernel and monthly market returns is -0.98 , whereas the conditional correlation between the two is, effectively, -1 . Thus, even at the monthly frequency, a conditional CAPM is highly accurate. In this respect our environment differs crucially from that in Berk et al. (1999) since, in their model, stock returns cannot be described using market returns as a single risk factor. This allows them to have variables, other than market beta, playing an independent role in predicting expected returns.

Figure 2 shows the behavior of the key economic quantities that determine firm-level betas against the state variable x . As expected, the optimal investment policy, $p(x)$, which, in equilibrium, equals the present value of cash flows from a new project of unit size, V^*/K , is increasing in x . Similarly, the market value per unit scale of a typical project, V/K , is also increasing in x . Given our calibration, assets in place account for about 75–80 percent of the total stock market value. This fraction is countercyclical since more new projects are adopted in good times. Finally, figure 2*f* confirms that the beta of growth options, β^g , is higher than that of assets in place, β^a , which, from (28), guarantees a negative (partial) correlation between firm size and firm beta.

Finally, it seems natural to examine the implications of our model for the relation between returns and book-to-market at the aggregate level before investigating this link in the cross section. Table 2 compares our results to those in Pontiff and Schall (1998). Panel A reports the means, standard deviations, and one- to five-year autocorrelations of the dividend yield and book-to-market ratio. While both means and standard deviations seem very similar, the book-to-market ratio is more persistent in our model. Panel B examines the performance of the book-to-market ratio as a predictor of stock market returns at monthly and annual frequencies. In both cases, our model produces somewhat lower, but statistically comparable, values for the slopes and the adjusted R^2 's.

TABLE 2
 BOOK-TO-MARKET AS A PREDICTOR OF MARKET RETURNS
 A. MEANS, STANDARD DEVIATIONS, AND AUTOCORRELATIONS

	Source	Mean	Standard Deviation					
			1 Year	2 Years	3 Years	4 Years	5 Years	
Dividend yield	Data	4.267	1.37	.60	.36	.26	.23	.25
	Model	6.407 (.321) [5.789 7.084]	.97 (.22) [.61 1.45]	.69 (.08) [.51 .82]	.46 (.14) [.17 .70]	.31 (.17) [-.05 .61]	.19 (.18) [-.16 .51]	.11 (.18) [-.22 .45]
Book-to-market	Data	.668	.23	.68	.43	.23	.08	.00
	Model	.584 (.052) [.495 .707]	.19 (.04) [.12 .28]	.88 (.03) [.81 .93]	.80 (.07) [.63 .89]	.73 (.09) [.48 .86]	.68 (.12) [.38 .84]	.64 (.13) [.31 .83]

B. REGRESSIONS ON BOOK-TO-MARKET

	DATA		MODEL	
	Slope	Adjusted R^2	Slope	Adjusted R^2
Monthly	3.02	.01	1.75 (.79) [.68 3.65]	.00 (.00) [.00 .01]
Annual	42.18	.16	19.88 (10.46) [6.57 46.09]	.04 (.04) [.00 .14]

NOTE.—This table examines the model's ability to match the empirical regularities in Pontiff and Schall (1998). Panel A reports means, standard deviations, and autocorrelations of dividend yield and book-to-market ratio, both from historical data and from simulations. The numbers in data rows are taken from the last two rows in panel A of table 1 of Pontiff and Schall's study. Panel B reports the properties of the regression of value-weighted market returns, both at a monthly and annual frequency, on one-period-lagged book-to-market. The columns labeled Data are taken from table 2 of Pontiff and Schall's study. In both panels, the numbers labeled Model report the statistics from 200 simulations, each of which has the same length as that of the data set used in Pontiff and Schall's study. The numbers in parentheses are standard deviations across 200 simulations, and the two numbers in brackets are 2.5th and 97.5th percentiles, respectively. All numbers, except autocorrelations and adjusted R^2 's, are in percentages.

IV. The Cross Section of Stock Returns

This section establishes our key quantitative results. After we outline our numerical procedure in subsections *A* and *B*, subsection *C* examines the model's implications for the relation between firm characteristics and stock returns. Subsection *D* concludes with a description of the conditional, or cyclical, properties of firm-level returns.

A. Calibration

To examine the cross-sectional implications of the model, we need to choose parameters for the stochastic process of the firm-specific productivity shocks, θ_ϵ and σ_ϵ . They are restricted by two considerations. First, we must generate empirically plausible levels of volatility of individual stock returns, which directly affects statistical inference about the relations between returns and firm characteristics. Second, we also want to match the observed cross-sectional correlation between (the logarithms of) firm value and the book-to-market ratio since, as we shall see below, this correlation is critical in determining the *univariate* relations between firm characteristics and returns.

Our goals are accomplished by setting $\theta_\epsilon = 0.50$ and $\sigma_\epsilon = 2.00$. These values imply an average annualized volatility of individual stock returns of approximately 27 percent (a number between the 25 percent reported by Campbell et al. [2001] and 32 percent reported by Vuolteenaho [2002]), while exactly matching the observed correlation between size and book-to-market ($-.26$) reported by Fama and French (1992).

The sign of the cross-sectional relation between the conditional market betas and firm characteristics depends on the aggregate-level variables $\beta_i^o - \tilde{\beta}_i^a$ and $\beta_i^a - \tilde{\beta}_i^a$ in (28). Given our parameter choices, the long-run average values of $\beta_i^o - \tilde{\beta}_i^a$ and $\beta_i^a - \tilde{\beta}_i^a$ are 0.67 and 0.21, respectively, thus guaranteeing a negative relation between the conditional market beta and firm size and a positive one between the conditional beta and book-to-market. Given the negative correlation between size and book-to-market, the signs of these partial regression coefficients will be preserved in univariate regressions, despite the omitted variable bias.

B. Simulation and Estimation

Our artificial panel is carefully constructed to replicate the procedures in Fama and French (1992). We start by constructing an artificial panel consisting of 360 months of observations for 2,000 firms. This is comparable to the panel of 2,267 firms for 318 months used in Fama and French's study. We adhere to Fama and French's timing conventions by

TABLE 3
PROPERTIES OF PORTFOLIOS FORMED ON SIZE

	PORTFOLIO											
	1A	1B	2	3	4	5	6	7	8	9	10A	10B
A. Historical Data												
Return	1.64	1.16	1.29	1.24	1.25	1.29	1.17	1.07	1.10	.95	.88	.90
β	1.44	1.44	1.39	1.34	1.33	1.24	1.22	1.16	1.08	1.02	.95	.90
$\log(V_t)$	1.98	3.18	3.63	4.10	4.50	4.89	5.30	5.73	6.24	6.82	7.39	8.44
$\log(B_t/V_t)$	-.01	-.21	-.23	-.26	-.32	-.36	-.36	-.44	-.40	-.42	-.51	-.65
B. Simulated Panel												
Return	.73	.72	.71	.70	.69	.70	.68	.67	.66	.64	.61	.55
β	1.05	1.05	1.03	1.02	1.01	1.01	1.00	.99	.97	.95	.89	.89
$\log(V_t)$	4.86	5.04	5.12	5.16	5.20	5.24	5.27	5.32	5.37	5.46	5.58	5.84
$\log(B_t/V_t)$	-.93	-.86	-.85	-.84	-.85	-.86	-.87	-.90	-.97	-1.09	-1.24	-1.49

NOTE.—At the end of June of each year t , 12 portfolios are formed on the basis of ranked values of size. Portfolios 2–9 cover corresponding deciles of the ranking variables. The bottom and top two portfolios (1A, 1B, 10A, and 10B) split the bottom and top deciles in half. The break points for the size portfolios are based on ranked values of size. Panel A is taken from Fama and French (1992, table 2, panel A). Panel B is constructed from the simulated panel. The average returns are the time-series averages of the monthly equal-weighted portfolio returns, in percentages. The terms $\log(V_t)$ and $\log(B_t/V_t)$ are the time-series averages of the monthly average values of these variables in each portfolio, and β is the time-series average of the monthly portfolio postranking betas.

matching accounting variables at the end of the calendar year $t - 1$ with returns from July of year t to June of year $t + 1$. Moreover, we also use the values of the firm's equity at the end of calendar year $t - 1$ to compute its book-to-market ratios for year $t - 1$ and use its market capitalization for June of year t as a measure of its size.⁷ In all cases we repeat the entire simulations 100 times and average our results across the simulations. Further details of our simulation procedure are summarized in Appendix B.

Some of our tests use estimates of market betas of stock returns, which are obtained using the empirical procedure detailed in Fama and French (1992). Essentially, their procedure consists of two steps. First, *preranking* betas for each firm and period are estimated on the basis of the previous 60 monthly returns. Second, for each month, stocks are grouped into 10 portfolios sorted by market value. Each portfolio is then further divided into 10 subportfolios by sorting stocks according to their preranking betas.⁸ *Postranking* betas are then estimated for each portfolio, and these betas are then allocated to *each* of the stocks within the portfolio. We shall refer to these betas as the Fama-French betas.

C. Size and Book-to-Market Effects

Tables 3 and 4 compare the summary statistics of our model with those reported by Fama and French (1992). We report the postranking average

⁷ Berk et al. (1999) use only a straightforward timing convention (one-period-lag values of explanatory variables) that does not agree with the definitions in Fama and French (1992).

⁸ Sometimes the top and bottom deciles are also divided in half.

TABLE 4
 PROPERTIES OF PORTFOLIOS FORMED ON BOOK-TO-MARKET

	PORTFOLIO											
	1A	1B	2	3	4	5	6	7	8	9	10A	10B
A. Historical Data												
Return	.30	.67	.87	.97	1.04	1.17	1.30	1.44	1.50	1.59	1.92	1.83
β	1.36	1.34	1.32	1.30	1.28	1.27	1.27	1.27	1.27	1.29	1.33	1.35
$\log(V)$	4.53	4.67	4.69	4.56	4.47	4.38	4.23	4.06	3.85	3.51	3.06	2.65
$\log(B/V)$	-2.22	-1.51	-1.09	-.75	-.51	-.32	-.14	.03	.21	.42	.66	1.02
B. Simulated Panel												
Return	.61	.65	.67	.70	.70	.71	.71	.71	.71	.70	.71	.71
β	.95	.98	1.01	1.02	1.02	1.02	1.03	1.02	1.02	1.02	1.02	1.02
$\log(V)$	5.54	5.30	5.18	5.11	5.10	5.09	5.10	5.10	5.12	5.13	5.14	5.16
$\log(B/V)$	-1.54	-1.29	-1.15	-1.05	-.98	-.92	-.87	-.83	-.78	-.72	-.66	-.59

NOTE.—At the end of June of each year t , 12 portfolios are formed on the basis of ranked values of book-to-market, measured by $\log(B/V)$. The preranking betas use five years of monthly returns ending in June of t . Portfolios 2–9 cover deciles of the ranking variables. The bottom and top two portfolios (1A, 1B, 10A, and 10B) split the bottom and top deciles in half. The break points for the book-to-market portfolios are based on ranked values of book-to-market equity. Panel A is taken from Fama and French (1992, table 4, panel A). Panel B is taken from the simulated panel. The average returns are the time-series averages of the monthly equal-weighted portfolio returns, in percentages. The terms $\log(V)$ and $\log(B/V)$ are the time-series averages of the monthly average values of these variables in each portfolio, and β is the time-series average of the monthly portfolio postranking betas.

returns for portfolios formed by a one-dimensional sort of stocks on firm size and book-to-market. Panel A is taken from Fama and French (1992) and panel B is computed on the basis of the simulated panels.

Since our model abstracts from inflation, the level of stock returns is naturally higher in panel A. In both cases, however, the pattern of stock returns in the model seems to match the evidence well. Similarly to the historical data, our simulated panels show a negative relation between average returns and firm value (table 3) and a positive relation with the book-to-market ratio (table 4).

Table 5 shows the results from the Fama and MacBeth (1973) regressions of stock returns on size, book-to-market, and the conditional market betas implied by our theoretical model. For each simulation, the slope coefficients are the time-series averages of the cross-sectional regression coefficients, and the t -statistics are these averages divided by the time-series standard deviations. We also report empirical findings of Fama and French (1992) and simulation results of Berk et al. (1999) in columns 1 and 2 of the same table. For completeness, figure 3 shows the histogram of the realized t -statistics across simulations.

Our first univariate regression shows that the logarithm of firm market value appears to contain useful information about the cross section of stock returns in our model. The relation between returns and size is significantly negative. Moreover, the average slope coefficient as well as the corresponding t -statistic are close to their empirical values reported by Fama and French (1992). Figure 3a also shows that the empirical

TABLE 5
EXACT REGRESSIONS

	Fama-French (1)	Berk et al. (2)	Benchmark (3)	High Variance (4)	Low Persistence (5)
$\log(V)$	-.15 (-2.58)	-.035 (-.956)	-.139 (-2.629)	-.172 (-3.016)	-.141 (-2.729)
$\log(B/V)$.50 (5.71)082 (1.955)	.107 (2.274)	.103 (2.341)
$\log(V)$	-.11 (-1.99)	-.093 (-2.237)	-.127 (-2.516)	-.156 (-2.875)	-.121 (-2.446)
$\log(B/V)$.35 (4.44)	.393 (2.641)	.045 (1.225)	.053 (1.261)	.052 (1.340)
β	-.37 (-1.21)	.642 (2.273)	1.048 (2.629)	1.193 (2.634)	1.050 (2.454)
$\log(V)$	-.17 (-3.41)	.053 (1.001)	.033 (.518)	.022 (.323)	.035 (.524)
β892 (2.933)	1.085 (3.337)	.859 (2.893)
$\log(B/V)$014 (.385)	.020 (.452)	.024 (.604)
β	.15 (.46)	.377 (1.542)	.916 (3.079)	1.115 (3.432)	.914 (3.106)

NOTE.—This table lists summary statistics for the coefficients and the t -statistics of Fama-MacBeth regressions using exact conditional β on the simulated panel sets. The dependent variable is the realized stock return, and independent variables are market beta, the logarithm of the market value, $\log(V)$, and the logarithm of the book-to-market ratio, $\log(B/V)$. Col. 1 gives the empirical results obtained by Fama and French (1992, table 3), using the historical returns of 2,267 firms over 318 months. Col. 2 gives the results obtained by Berk et al. (1999). Col. 3 reports the regression results for our model under benchmark parameterization in Sec. IIIA. Col. 4 reports the results from the model with the calibrated parameter values $\theta_t = 0.50$ and $\sigma_t = 2.50$ such that the average individual volatility is 30 percent, which is higher than the benchmark case of 27 percent. Col. 5 reports the results from the model with the calibrated parameter value $\theta_t = 0.40$ such that the persistence level is now lower. The regression coefficients are in percentage terms. The numbers in parentheses are t -statistics. The coefficients in the columns are in percentage terms. The numbers in parentheses are their corresponding t -statistics. Both coefficients and t -statistics are averaged across 100 simulations.

value of the t -statistic is well within the body of realizations produced by the model.

The second univariate regression confirms the importance of the book-to-market ratio in explaining the cross-sectional properties of stock returns. While both our slope coefficient and t -statistic are smaller than the values obtained by Fama and French (1992), our estimates are positive and, as figure 3*a* shows, the coefficient of book-to-market is often quite significant at traditional levels. In Section IIC, we argued that the book-to-market ratio in our model is related to expected returns because it is a proxy for firm productivity; that is, firms with a higher book-to-market ratio tend to be less productive and therefore have higher systematic risk. Figure 4 shows that, in our model, a negative relation also exists between the book-to-market ratio and firm profitability, defined as the ratio of profits (output) to book value. Firms with a low book-

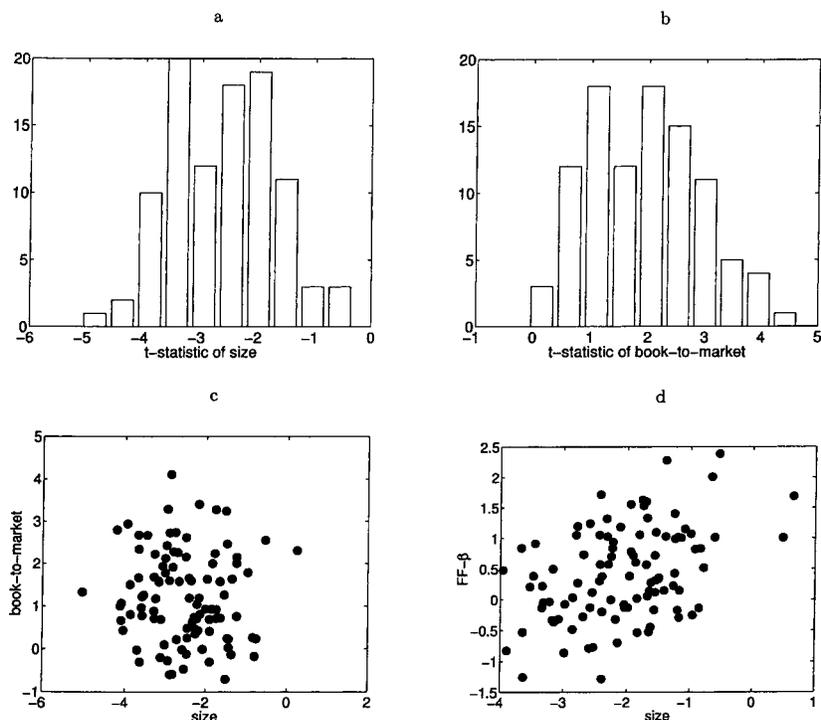


FIG. 3.—Size and book-to-market in cross-sectional regressions. *a*, The histogram of t -statistics of univariate regressions of returns on size. *b*, The histogram of t -statistics of univariate regressions of returns on book-to-market across 100 simulations. *c*, The scatter plot of t -statistics on size and book-to-market. *d*, The scatter plot of t -statistics on size and Fama-French (FF) beta in a joint regression of returns.

to-market are more productive than firms with a high book-to-market both before and after the portfolio formation date, with the difference in productivity declining over time. This pattern is also qualitatively consistent with the empirical results reported in Fama and French (1995, fig. 2).

Regressing returns on size and book-to-market jointly, we find that, on average, our coefficients have the same signs as in Fama and French (1992) and Berk et al. (1999). While our average size slope and the corresponding t -statistic are close to the empirical values, the average slope on book-to-market is again smaller than in Fama and French (1992). Figure 3*c* illustrates the range of t -statistics in a joint regression of returns on size and book-to-market. Each point corresponds to a realization of two t -statistics obtained in a single simulation. It is clear that, while the observed t -statistic on the size variable is comparable to

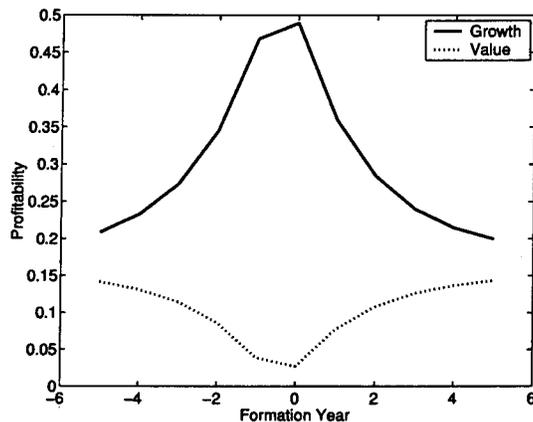


FIG. 4.—Value factor in earnings. This figure illustrates the relation between the book-to-market ratio and firm profitability in the simulated data. It shows the 11-year evolution of profitability for book-to-market portfolios. Growth (value) indicates the portfolio containing firms in the bottom (top) 30 percent of the values of book-to-market ratios. Profitability (or return on book equity) is measured by $(\Delta B_j + D_j)/B_{j,t-1}$, where B_j denotes the book value of equity and D is the dividend payout. Thus profitability equals the ratio of common equity income for the fiscal year ending in calendar year t to the book value of equity for year $t-1$. The profitability of a portfolio is defined as the sum of $\Delta B_j + D_j$ for all firms j in a portfolio divided by the sum of $B_{j,t-1}$; thus it is the return on book equity by merging all firms in the portfolio. For each portfolio formation year t , the ratios of $(\Delta B_{t+i} + D_{t+i})/B_{t+i-1}$ are calculated for year $t+i$, $i = -5, \dots, 5$. The ratio for year $t+i$ is then averaged across portfolio formation years. Time 0 in the horizontal axis is the portfolio formation year.

typical realizations produced by the model, the t -statistic on book-to-market is usually lower than that in Fama and French (1992).

These first three regressions in table 5 conform to the intuition that size and book-to-market are related to systematic risks of stock returns and therefore have explanatory power in the cross section. The fourth row of table 5 shows, however, that when we control for market beta, both the average coefficient on size and the corresponding t -statistic are close to zero. Within our theoretical framework, firm characteristics add no explanatory power to the conditional market betas of stock returns. This is not surprising since the market betas are sufficient statistics for instantaneous expected returns in our model. As shown in Section III, even at monthly frequency, the market portfolio is almost perfectly correlated with the pricing kernel.

To reconcile our results with the poor empirical performance of beta, one must take into account the fact that we have been using exact conditional betas, which are not observable in practice. Instead, betas

must be estimated, which leaves room for measurement error.⁹ Given the relation between beta and firm characteristics in (28), this measurement error in beta will also have an effect of creating a role for size and book-to-market as predictors of expected returns.

To illustrate the impact of beta mismeasurement, we now apply the Fama and French (1992) estimation procedure to our simulated data. Table 6 provides preliminary evidence on the relation between the Fama and French beta and average returns. As in the data, we find that after stocks have been sorted by size, variation in beta sort produces very little variation in average returns.

As table 7 shows, using the Fama-French beta significantly changes our results. Now, beta is, on average, statistically insignificant whereas size remains both negative and significant even in a joint regression with beta. The scatter plot in figure 3d shows that the *t*-statistic on the Fama-French beta is usually far below 1.96, whereas the coefficient on size often appears significant.

Table 8 presents a measure of the noise in the construction of the Fama-French beta. It shows the average correlation matrix (standard errors included) between the true conditional betas, Fama-French betas, size, and book-to-market. It is easy to see that while the exact conditional beta is highly negatively correlated with size, the correlation with the Fama-French beta is much lower. Not surprisingly then, size serves as a more accurate measure of systematic risk than Fama-French beta and hence outperforms it in a cross-sectional regression.

The relation between expected returns, firm size, and market beta in our model is drastically different from that in the partial equilibrium model of Berk et al. (1999), who report that in a joint regression, firm size enters with a positive coefficient, on average; the loading on the Fama-French beta is positive and significant. Both in our model and in the Berk et al. model, the firm size proxies for the relative value of the firm's growth options. However, while in our model growth options are driven by the same risk factor as the assets in place and are relatively more risky, in the model of Berk et al., the growth options load only on the interest rates and therefore have a relatively low risk premium. Such a difference in the properties of growth options could explain why the two models have very different implications for the joint behavior of returns, firm size, and the market beta.

⁹ Potential sources of errors are, among others, the fact that the market proxy used in estimation is not the mean-variance efficient portfolio (Roll 1977) or the econometric methods employed in estimation do not adequately capture the *conditional* nature of the pricing model (e.g., Ferson, Kandel, and Stambaugh 1987; Jagannathan and Wang 1996; Ferson and Harvey 1999; Campbell and Cochrane 2000; Lettau and Ludvigson 2001).

TABLE 6
AVERAGE RETURNS FOR PORTFOLIOS FORMED ON SIZE (DOWN) AND THEN BETA (ACROSS)

	All	Low- β	β -2	β -3	β -4	β -5	β -6	β -7	β -8	β -9	High- β
A. Average Monthly Returns (Percent) from Fama and French (1992)											
All	1.25	1.34	1.29	1.36	1.31	1.33	1.28	1.24	1.21	1.25	1.14
Small market equity	1.52	1.71	1.57	1.79	1.61	1.50	1.50	1.37	1.63	1.50	1.42
Market equity 2	1.29	1.25	1.42	1.36	1.39	1.65	1.61	1.37	1.31	1.34	1.11
Market equity 3	1.24	1.12	1.31	1.17	1.70	1.29	1.10	1.31	1.36	1.26	.76
Market equity 4	1.25	1.27	1.13	1.54	1.06	1.34	1.06	1.41	1.17	1.35	.98
Market equity 5	1.29	1.34	1.42	1.39	1.48	1.42	1.18	1.13	1.27	1.18	1.08
Market equity 6	1.17	1.08	1.53	1.27	1.15	1.20	1.21	1.18	1.04	1.07	1.02
Market equity 7	1.07	.95	1.21	1.26	1.09	1.18	1.11	1.24	.62	1.32	.76
Market equity 8	1.10	1.09	1.05	1.37	1.20	1.27	.98	1.18	1.02	1.01	.94
Market equity 9	.95	.98	.88	1.02	1.14	1.07	1.23	.94	.82	.88	.59
Large market equity	.89	1.01	.93	1.10	.94	.94	.89	1.03	.71	.74	.56
B. Average Monthly Returns (Percent) from Simulated Panel											
All	.67	.67	.68	.67	.68	.68	.68	.67	.68	.68	.67
Small market equity	.72	.72	.72	.72	.72	.73	.72	.72	.73	.72	.72
Market equity 2	.71	.70	.71	.71	.71	.70	.72	.71	.70	.71	.70
Market equity 3	.70	.70	.70	.70	.71	.69	.70	.70	.69	.71	.70
Market equity 4	.69	.69	.69	.69	.71	.70	.70	.67	.70	.69	.68
Market equity 5	.70	.70	.72	.70	.70	.71	.71	.69	.70	.69	.68
Market equity 6	.68	.64	.68	.68	.67	.70	.69	.68	.69	.69	.70
Market equity 7	.67	.65	.66	.65	.68	.68	.68	.67	.65	.69	.65
Market equity 8	.66	.64	.67	.65	.67	.68	.66	.66	.64	.67	.65
Market equity 9	.64	.61	.65	.61	.65	.63	.63	.64	.66	.64	.65
Large market equity	.58	.61	.56	.55	.57	.55	.63	.58	.61	.59	.56

NOTE.—Panel A is identical to panel A of table 1 in Fama and French (1992), in which the authors report average returns for 100 size- β portfolios using all New York Stock Exchange, American Stock Exchange, and NASDAQ stocks from July 1963 to December 1990 that meet certain Center for Research in Security Prices-Compustat data requirements. Panel B is produced using our simulated panel data set. The portfolio-sorting procedure is identical to that used in Fama and French's study. In particular, portfolios are formed yearly. The break points for the size deciles are determined in June of year t using all the stocks in the panel. All the stocks are then allocated to the 10 size portfolios using the break points. Each size decile is further subdivided into 10 beta portfolios using preranking betas of individual stocks, estimated with five years of monthly returns ending in June of year t . The equal-weighted monthly returns on the resulting 100 portfolios are then calculated for July of year t to June of year $t+1$. The preranking betas are the sum of the slopes from a regression of monthly returns on the current and prior month's market returns. The average return is the time-series average of the monthly equal-weighted portfolio returns (percent). The All column shows statistics for equal-weighted size-decile (market equity) portfolios and the All rows show statistics for equal-weighted portfolios of the stocks in each beta group.

Sensitivity Analysis

Columns 4 and 5 of tables 5 and 7 report the effects of alternative choices for the parameters, θ_ϵ and σ_ϵ , governing the cross-sectional properties of stock returns. Column 4 in these tables looks at the effects of increasing the cross-sectional dispersion of stock returns to 30 percent, which corresponds to a value for σ_ϵ of 2.50. Column 5 studies the effects of changing the persistence of the idiosyncratic productivity shocks by raising the value of θ_ϵ to 0.4. In both cases it is easy to see that our main

TABLE 7
FAMA-FRENCH REGRESSIONS

	Fama-French (1)	Berk et al. (2)	Benchmark (3)	High Variance (4)	Low Persistence (5)
$\log(V_t)$	-.15 (-2.58)	-.035 (-.956)	-.139 (-2.629)	-.172 (-3.016)	-.141 (-2.729)
$\log(B_t/V_t)$.50 (5.71)082 (1.955)	.107 (2.274)	.103 (2.341)
$\log(V_t)$	-.11 (-1.99)	-.093 (-2.237)	-.127 (-2.516)	-.156 (-2.875)	-.121 (-2.446)
$\log(B_t/V_t)$.35 (4.44)	.393 (2.641)	.045 (1.225)	.053 (1.261)	.052 (1.340)
β	-.37 (-1.21)	.642 (2.273)	.133 (.429)	.178 (.590)	.214 (.727)
$\log(V_t)$	-.17 (-3.41)	.053 (1.001)	-.121 (-2.057)	-.151 (-2.298)	-.108 (-1.821)
β	.15 (.46)	.377 (1.542)	.590 (2.158)	.721 (2.472)	.605 (2.367)

NOTE.—This table lists summary statistics for the coefficients and the t -statistics of Fama-MacBeth regressions using the estimated Fama-French β on the simulated panel sets. See also the note to table 5.

results appear to be robust. In all cases, both the signs and significance of all the coefficients are preserved.

D. Business Cycle Properties

The theoretical characterization of stock prices and systematic risk, as given by (27) and (28), highlights the fact that the properties of the cross section of stock prices and stock returns depend on the current state of the economy. This dependence is captured by the economywide variables $p(x_t)$, $\tilde{p}(x_t)$, and V_t^o and their market betas. Thus our model also gives rise to a number of predictions about the variation of the

TABLE 8
CROSS-SECTIONAL CORRELATIONS

	True β	Fama-French β	$\log(B_t/V_t)$	$\log(V_t)$
True β	1	.598 (.028)	.324 (.022)	-.764 (.012)
Fama-French β		1	.270 (.031)	-.758 (.036)
$\log(B_t/V_t)$			1	-.262 (.019)
$\log(V_t)$				1

NOTE.—We calculate the cross-sectional correlations of exact conditional beta, Fama-French beta, book-to-market, and size for every simulated panel every month and then report the average correlations across 100 simulations. The numbers in parentheses are cross-simulation standard deviations.

cross section of stock prices and returns over the business cycle. These properties of the cross section of stock returns may have important implications for optimal dynamic portfolio choice.

Firm Characteristics

To help understand the relation between the cross section of firm characteristics and the business cycle, we first characterize the cross-sectional dispersion of firm market values. To this end, let $\text{Var}(h)$ denote the variance of the cross-sectional distribution of a firm-specific variable h . According to our characterization of firm market value (27), it follows immediately that

$$\begin{aligned} \text{Var}\left(\frac{V_t}{V_t}\right) &= \left[\frac{\tilde{p}(x_t)K_t}{V_t}\right]^2 \text{Var}\left[\sum_{i \in \mathcal{I}_t} (\epsilon_{it} - 1) \frac{k_i}{K_t}\right] \\ &\quad + \left[\frac{p(x_t)K_t}{V_t}\right]^2 \text{Var}\left[\sum_{i \in \mathcal{I}_t} \frac{k_i}{K_t}\right]. \end{aligned} \quad (30)$$

The right-hand side of (30) captures the cross-sectional dispersion of relative firm size. This dispersion can be attributed to (i) the cross-sectional variation of project-specific productivity shocks ϵ_{it} as well as project-specific and firm-specific production scale and (ii) economywide variables $p(x_t)$, $\tilde{p}(x_t)$, and K_t/V_t .

The contribution of the firm heterogeneity, captured by

$$\text{Var}\left[\sum_{i \in \mathcal{I}_t} \frac{k_i}{K_t}\right]$$

and

$$\text{Var}\left[\sum_{i \in \mathcal{I}_t} (\epsilon_{it} - 1) \frac{k_i}{K_t}\right],$$

is clearly path-dependent in theory, since the scale of new projects depends on the current aggregate scale of production K_t . Intuitively, however, this dependence is fairly low when the average lifetime of individual projects is much longer than the average length of a typical business cycle.¹⁰

It falls then on the aggregate components to determine the cross-sectional variance in market value. Given the properties of our environment, it is easy to see that this implies that the cross-sectional dispersion of firm size is countercyclical; that is, it expands in recessions

¹⁰ Note that the average project life is about $1/\delta = 25$ years, given our calibration.

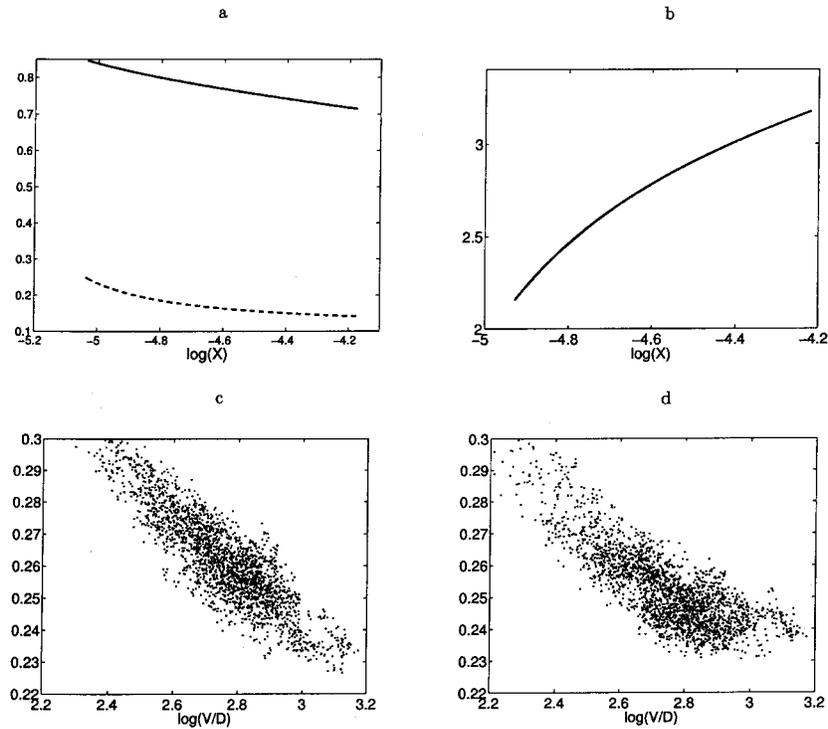


FIG. 5.—Business cycle properties (I). This figure illustrates the business cycle properties of some aggregate and cross-sectional variables. *a*, $p(x)K/V$ (the solid line) and $\tilde{p}(x)K/V$ (the dashed line) plotted as functions of x . *b*, Log of the price-dividend ratio, $\log(V/D)$, as a function of $\log(X)$. *c*, Size ($\log[V_i]$) dispersion as a function of $\log(V/D)$. *d*, Dispersion of book-to-market ($\log[B_i/V_i]$) as a function of $\log(V/D)$.

and becomes compressed in expansions. We can see this by looking at figure 2*d*. Since the market betas of $p(x_i)$ and $\tilde{p}(x_i)$ are less than one, the ratios $p(x_i)K_i/V_i$ and $\tilde{p}(x_i)K_i/V_i$ should be negatively related to the state variable x_t . Figure 5 confirms this finding.

To quantify this relation, we simulate our artificial economy over a 200-year period and compute the cross-sectional standard deviation of the logarithm of firm values and book-to-market ratios on a monthly basis. Since the state variable x_t is not observable empirically, we choose to capture the current state of the economy by the price-to-dividend ratio of the aggregate stock market.¹¹

Figure 5 presents scatter plots of the cross-sectional dispersion of firm characteristics against the logarithm of the aggregate price-dividend ratio. In both cases the relation is clearly negative. Note that cross-

¹¹ In the model, the unconditional correlation between x_t and $\log(V/D)$ is 98.8 percent.

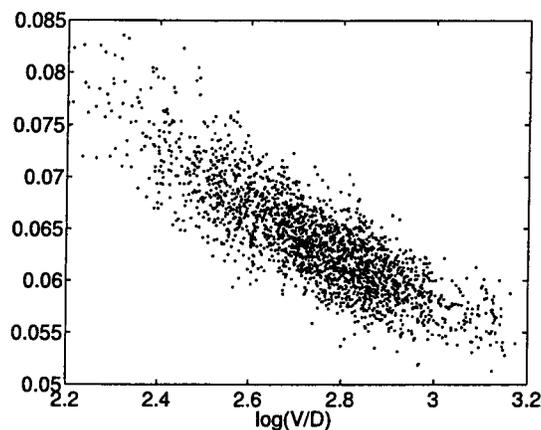


FIG. 6.—Return dispersion over the business cycle. This figure plots the cross-sectional dispersion of the firm-level stock returns, $RD_t = \sqrt{\text{Var}(R_{jt})}$, against $\log(V_t/D_t)$.

sectional dispersion is not a simple function of the state variable partially because we are using a finite number of firms and projects in our simulation; therefore, our theoretical relations hold only approximately. Moreover, as suggested by the theoretical argument above, such relations are inherently history-dependent.

Stock Returns

Next we study how the cross-sectional distribution of actual stock returns depends on the state of the aggregate economy. First, we analyze the degree of dispersion of returns, $RD_t = \sqrt{\text{Var}(R_{jt})}$, where R_{jt} denotes monthly returns on individual stocks. We construct a scatter plot of RD_t versus contemporaneous values of the logarithm of the aggregate price-dividend ratio.

According to figure 6, our model predicts a negative contemporaneous relation between return dispersion and the price-dividend ratio. This can be attributed to the countercyclical nature of both aggregate return volatility, as shown in figure 7*a*, and the dispersion in conditional market beta, as shown in figure 7*b*.

Since investment in our model is endogenously procyclical, an increase in aggregate productivity shock is accompanied by an increase in the rate of investment and hence a higher growth rate of the scale of production, as well as an increase in stock prices. On the other hand, since investment is irreversible, the scale of production cannot be easily

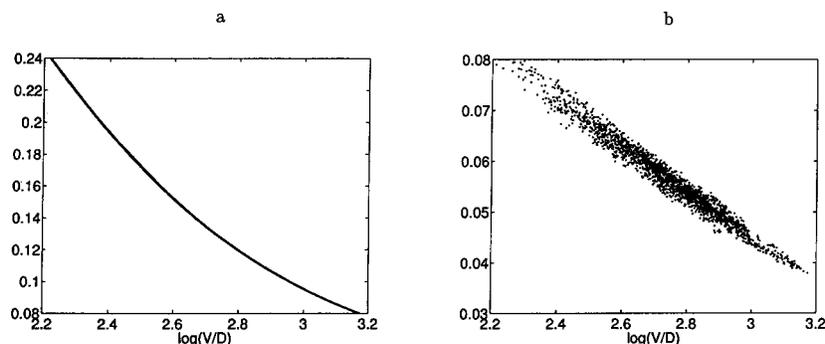


FIG. 7.—Business cycle properties (II). This figure illustrates the business cycle properties of some aggregate and cross-sectional variables. *a*, Conditional stock market return volatility plotted as a function of the log price-dividend ratio, $\log(V/D)$. *b*, Cross-sectional dispersion of the firm-level market betas, $\sqrt{\text{Var}(\beta_i)}$, plotted against $\log(V/D)$.

reduced during periods of low aggregate productivity, increasing volatility of stock prices.¹²

The countercyclical dispersion of conditional betas follows from the characterization of the systematic risk of stock returns (28) and the pattern observed in figure 2*d*. During business cycle peaks, the dispersion of aggregate betas, that is, β_i^a , β_i^a , and β_i^a , is relatively low, contributing to lower dispersion of firm-level market betas. This effect is then reinforced by the countercyclical behavior of dispersion of firm characteristics. In this respect our model matches the empirical regularity pointed out by Chan and Chen (1988, n. 6), that the cross-sectional spectrum of conditional market betas of size-sorted portfolios contracts during business cycle booms and expands during business cycle troughs.

An interesting empirical finding by Stivers (2001) is the ability of return dispersion to forecast future aggregate return volatility, even after one controls for the lagged values of market returns. We conduct a similar experiment within our model by simulating monthly stock returns and regressing absolute values of aggregate market returns on lagged values of return dispersion and market returns. As in Stivers' study, we allow for different slope coefficients depending on the sign of lagged market returns. As shown in table 9, return dispersion retains significant explanatory power even after we control for market returns in the regression. The reason is that lagged market returns provide only a noisy proxy for the current state of the economy. At the same time, return dispersion contains independent information about the current

¹² Qualitatively, the impact of the irreversibility on conditional volatility of stock returns in our model is similar to that in Kogan (2000, 2001).

TABLE 9
CROSS-SECTIONAL RETURN DISPERSION AS A PREDICTOR OF MARKET VOLATILITY

	COEFFICIENT				JOINT	JOINT	R^2 (%)
	b_1	b_2	c_1	c_2	$b_1=0$ $b_2=0$	$c_1=0$ $c_2=0$	
A. Results from Stivers (2001)							
Full model	.365 (3.61)	.111 (1.40)	-.157 (-2.94)	.221 (1.84)	10.08 (.000)	2.69 (.069)	10.45
B. Simulation Results							
Full model	.918 (3.408)	.016 (.233)	.019 (.285)	.008 (.059)	8.467 (.035)	.949 (.534)	3.72

NOTE.—This table illustrates the intertemporal relation between market volatility and the lagged cross-sectional return dispersion (RD). The volatility is measured by the absolute value of the market excess return. Variations of the following model are estimated as follows:

$$|R_t^e| = a + b_1 RD_{t-1} + b_2 1_{|R_{t-1}^e|} RD_{t-1} + c_1 |R_{t-1}^e| + c_2 1_{|R_{t-1}^e|} |R_{t-1}^e| + \epsilon_t$$

where $|R_t^e|$ is the absolute value of the market excess return, RD_t is the cross-sectional standard deviation of the individual stock returns, $1_{|R_{t-1}^e|}$ is a dummy variable that equals one when the market excess return is negative and zero otherwise, and ϵ_t is the residual. All t -statistics are adjusted with respect to heteroscedasticity and autocorrelation using the Newey-West procedure. For the F -test on joint restrictions, the p -values are in parentheses. Panel A is taken from Stivers (2001), which uses 400 firm returns from July 1962 to December 1995. Panel B is generated as the average coefficients and statistics across repeated simulations.

state of the economy and hence the conditional stock market volatility, as shown in figure 6.

Conditional Size and Book-to-Market Effects

The fact that dispersion of returns on individual stocks in our model changes countercyclically suggests that the size and book-to-market effects analyzed in subsection C are also conditional in nature. To capture this cyclical behavior of cross-sectional patterns in returns and its implications for dynamic portfolio allocation, we analyze the conditional performance of alternative size- and value-based strategies. Specifically, we simulate 1,000 years of monthly individual stock returns and then form zero-investment portfolios by taking a long position in bottom-size-decile stocks and a short position in top-size-decile stocks, as sorted by size, with monthly rebalancing. We also construct alternative portfolios by doing the opposite for book-to-market deciles. We then regress portfolio returns on the logarithm of the aggregate price-dividend ratio.

Our model predicts an average annualized value (book-to-market) premium of 1.47 percent and an average annualized size premium of 1.62 percent. Moreover, both strategies exhibit significant countercyclical patterns in their expected returns. In particular, we find that a 10 percent decline in the log price-dividend ratio below its long-run mean implies approximately a 25 percent and 6 percent increase in expected

returns on the size and book-to-market strategies, respectively, measured as a fraction of their long-run average returns.

V. Conclusion

This paper analyzes a general equilibrium production economy with heterogeneous firms. In the model, the cross section of stock returns is explicitly related to firm characteristics such as size and book-to-market. Firms differ in the share of their total market value derived from their assets, as opposed to future growth opportunities, which is captured by their characteristics. Since these two components of firm value have different market risk, firm characteristics are closely related to market beta.

To the best of our knowledge, our paper is the first to explain the cross section of stock returns from a general equilibrium perspective. Our model demonstrates that size and book-to-market can explain the cross section of stock returns because they are correlated with the true conditional beta. We also provide an example of how empirically estimated beta can perform poorly relative to firm characteristics because of measurement errors.

Our model also gives rise to a number of additional implications for the cross section of returns. In this paper, we focus on the business cycle properties of returns and firm characteristics. Our results appear consistent with the limited existing evidence and provide a natural benchmark for future empirical studies.

Appendix A

Proofs and Technical Results

A. Proof of Proposition 1

The value of an ongoing project i is the present value of the future stream of cash flows, $e^{x_{i,t+s}}\epsilon_{i,t+s}k_i$, taking into account that the project can expire at the random rate δ . Hence

$$\begin{aligned} P(x_i, \epsilon_{i,t}, k_i) &= E_t \left[\int_0^{\infty} e^{-\delta s} M_{t,t+s} (e^{x_{i,t+s}} \epsilon_{i,t+s} k_i) ds \right] \\ &= k_i \left[\int_0^{\infty} e^{-\delta s} E_t [M_{t,t+s} e^{x_{i,t+s}}] E_t [\epsilon_{i,t+s}] ds \right], \end{aligned}$$

where the equality follows from mutual independence of X_t and $\epsilon_{i,t}$. Given (2), it follows that

$$E_t [\epsilon_{i,t+s}] = \epsilon_{i,t} e^{-\theta \epsilon s} + (1 - e^{-\theta \epsilon s}),$$

which implies that

$$\begin{aligned}
 P(x_t, \epsilon_{it}, k_t) &= k_i \left\{ \int_0^\infty e^{-\delta s} E_t[M_{t,t+s} e^{x_{t+s}}] [\epsilon_{it} e^{-\theta s} + (1 - e^{-\theta s})] ds \right\} \\
 &= k_i \left\{ \int_0^\infty e^{-\delta s} E_t[M_{t,t+s} e^{x_{t+s}}] ds + \int_0^\infty e^{-(\delta+\theta)s} E_t[M_{t,t+s} e^{x_{t+s}}] (\epsilon_{it} - 1) ds \right\} \\
 &= k_i [p(x_t) + \tilde{p}(x_t)(\epsilon_{it} - 1)],
 \end{aligned}$$

where $p(x_t)$ and $\tilde{p}(x_t)$ depend only on x_t since $M_{t,t+s}$ is a function of x_t and x_{t+s} and x_t is a univariate Markov process.

B. Proof of Proposition 2

Equation (17) is the resource constraint (16). As we have seen in Section IIA, the optimal firm investment policy $p(x)$ is defined by

$$p(x_t) = E_t \left[\int_0^\infty e^{-\lambda s} \left(\frac{C_t}{C_{t+s}} \right)^\gamma (e^{-\delta s} e^{x_{t+s}}) ds \right], \quad (\text{A1})$$

where we now impose that optimal consumption decisions are used in determining the pricing kernel in equilibrium. When we use the resource constraint (16) and the accumulation equation (9), it follows that

$$\begin{aligned}
 p(x_t) &= (C_t)^\gamma E_t \left[\int_0^\infty e^{-(\lambda+\delta)s} \frac{e^{x_{t+s}}}{(C_{t+s}/K_{t+s})^\gamma K_{t+s}^\gamma} ds \right] \\
 &= (C_t)^\gamma E_t \left[\int_0^\infty e^{-(\lambda+\delta)s} \frac{e^{x_{t+s}}}{\{e^{x_{t+s}} - \frac{1}{2}z[p(x_{t+s})]^2\} K_t^\gamma \exp \left[\int_0^s -\gamma\delta + \gamma z p(x_t) d\tau \right]} ds \right] \\
 &= \{e^{x_t} - \frac{1}{2}z[p(x_t)]^2\}^\gamma \phi(x_t),
 \end{aligned}$$

where the Feynman-Kac theorem implies then that $\phi(x)$ satisfies the differential equation (see, e.g., Duffie 1996, app. E):

$$[\lambda + (1 - \gamma)\delta + \gamma z p(x)]\phi(x) - \mathcal{L}[\phi(x)] - \frac{\exp(x)}{\{e^x - \frac{1}{2}z[p(x)]^2\}^\gamma} = 0.$$

C. Proof of Proposition 3

Let $m_t = (C_t)^{-\gamma}$ and $M_{t,t+s} = e^{-\lambda s} m_{t+s} / m_t$ and by Ito's lemma,

$$\begin{aligned} E_t[M_{t,t+dt} - 1] &= E_t \left[\frac{\partial M}{\partial s} \Big|_{s=0} dt + \frac{\partial m_t}{m_t \partial C_t} dC_t + \frac{1}{2} \frac{\partial^2 m_t}{\partial (C_t)^2} (dC_t)^2 \right] \\ &= -\lambda m_t dt - \frac{\gamma}{C_t} m_t E_t[dC_t] + \frac{1}{2} \frac{\gamma(\gamma+1)}{(C_t)^2} m_t E_t[dC_t]^2. \end{aligned}$$

Another application of Ito's lemma yields

$$\begin{aligned} E_t[dC_t] &= \frac{C(x_p, K_t)}{K_t} dK_t + K_t E_t \left[d \frac{C(x_p, K_t)}{K_t} \right] \\ &= C(x_p, K_t) [z\phi(x_t) - \delta] dt + \mathcal{A}[C(x_p, K_t)] dt, \\ (dC_t)^2 &= \left[\frac{\partial^2 C(x_p, K_t)}{\partial x_t^2} \right]^2 \sigma_x^2, \end{aligned}$$

where

$$\mathcal{A}[C(x, K)] \equiv \theta_x(x - \bar{x}) \frac{\partial C(x, K)}{\partial x} + \frac{1}{2} \frac{\partial^2 C(x, K)}{\partial x^2} \sigma_x^2.$$

As a result,

$$r_t = \lambda + \gamma [z\phi(x_t) - \delta] + \gamma \frac{\mathcal{A}[C(x_p, K_t)]}{C(x_p, K_t)} - \frac{1}{2} \gamma(\gamma+1) \sigma_x^2 \left[\frac{\partial \ln C(x_p, K_t)}{\partial x_t} \right]^2.$$

Now, the value of the aggregate stock market, V_p can be computed as

$$\begin{aligned} V_t &= E_t \left[\int_0^\infty M_{t,t+s} D_{t+s} ds \right] \\ &= E_t \left[\int_0^\infty e^{-\lambda s} \left(\frac{C_t}{C_{t+s}} \right)^\gamma C_{t+s} ds \right] \\ &= (C_t)^\gamma E_t \left[\int_0^\infty e^{-\lambda s} \left(\frac{C_{t+s}}{K_{t+s}} \right)^{1-\gamma} K_{t+s}^{1-\gamma} ds \right] \\ &= \left(\frac{C_t}{K_t} \right)^\gamma K_t E_t \left[\int_0^\infty e^{-\lambda s} \left(\frac{C_{t+s}}{K_{t+s}} \right)^{1-\gamma} \exp \left[\int_0^s -(1-\gamma)\delta + (1-\gamma)z\phi(x_\tau) d\tau \right] ds \right] \\ &= \{e^{x_t} - \frac{1}{2} z[\phi(x_t)]^2\}^\gamma \psi(x_t) K_p \end{aligned}$$

where $\psi(x)$ satisfies the following differential equation:

$$\lambda\psi(x) = \{e^{x_i} - \frac{1}{2}z[p(x)]^2\}^{1-\gamma} + (1-\gamma)[zp(x) - \delta]\psi(x) \\ - \theta_x(x - \bar{x})\psi'(x) + \frac{1}{2}\sigma_x^2\psi''(x).$$

Appendix B

Some Details of Computation

We use a finite, large number of firms in the numerical implementation. While the number of firms is fixed, the total number of projects in the economy is time-varying and stationary. We let the scale of new projects be proportional to the aggregate production scale in the economy, which ensures stationarity of the cross-sectional distribution of the number of projects per firm. Thus $k_i = K_i/\varphi$, where the constant φ controls the long-run average number of projects in the economy, N^* . On average, projects expire at the total rate δN^* . The arrival rate of new projects is $zp(x_i)\varphi$. Therefore, φ is defined from the equation $zE[p(x_i)]\varphi = \delta N^*$.

In the simulation, time increment is discrete. The unit costs of new projects are spaced out evenly over the interval $(0, p(x_i)]$. The investment of an individual firm at time t is computed as the total amount the firm spends on its new projects at time t . The dividend paid out by a given firm during period t is defined as the difference between the cash flows generated by the firm's existing projects and its investment. Finally, the individual firm's book value is measured as the cumulative investment cost of the firm's projects that remain active at time t .

In our simulation, we first generate 200 years' worth of monthly data to allow the economy to reach steady state. After that, we repeatedly simulate a 420-month panel data set consisting of the cross-sectional variables (360 months of data constitute the main panel, and 60 extra months are used for preranking β estimation).

References

- Berk, Jonathan B. "A Critique of Size-Related Anomalies." *Rev. Financial Studies* 8 (Summer 1995): 275–86.
- Berk, Jonathan B.; Green, Richard C.; and Naik, Vasant. "Optimal Investment, Growth Options, and Security Returns." *J. Finance* 54 (October 1999): 1553–1607.
- Bossaerts, Peter, and Green, Richard C. "A General Equilibrium Model of Changing Risk Premia: Theory and Tests." *Rev. Financial Studies* 2 (Winter 1989): 467–93.
- Breeden, Douglas T. "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities." *J. Financial Econ.* 7 (September 1979): 265–96.
- Campbell, John Y. "Asset Pricing at the Millennium." *J. Finance* 55 (August 2000): 1515–67.
- . "Consumption-Based Asset Pricing." In *Handbook of the Economics of Finance*, vol. 1B, edited by George M. Constantinides, Milton Harris, and Rene M. Stulz. Amsterdam: North-Holland, 2003 (in press).
- Campbell, John Y., and Cochrane, John H. "Explaining the Poor Performance

- of Consumption-Based Asset Pricing Models." *J. Finance* 55 (December 2000): 2863–78.
- Campbell, John Y.; Lettau, Martin; Malkiel, Burton G.; and Xu, Yexiao. "Have Individual Stocks Become More Volatile? An Empirical Exploration of Idiosyncratic Risk." *J. Finance* 56 (February 2001): 1–43.
- Campbell, John Y.; Lo, Andrew W.; and MacKinlay, A. Craig. *The Econometrics of Financial Markets*. Princeton, N.J.: Princeton Univ. Press, 1997.
- Chan, K. C., and Chen, Nai-Fu. "An Unconditional Asset-Pricing Test and the Role of Firm Size as an Instrumental Variable for Risk." *J. Finance* 43 (June 1988): 309–25.
- Cochrane, John H. "Production-Based Asset Pricing and the Link between Stock Returns and Economic Fluctuations." *J. Finance* 46 (March 1991): 209–37.
- . "A Cross-Sectional Test of an Investment-Based Asset Pricing Model." *J.P.E.* 104 (June 1996): 572–621.
- . "New Facts in Finance." *Fed. Reserve Bank Chicago Econ. Perspectives* 23 (3d quarter 1999): 36–58.
- Coleman, Wilbur J., II. "Behavior of Interest Rates in a General Equilibrium Multisector Model with Irreversible Investment." *Macroeconomic Dynamics* 1, no. 1 (1997): 206–27.
- Duffie, Darrell. *Dynamic Asset Pricing Theory*. 2d ed. Princeton, N.J.: Princeton Univ. Press, 1996.
- Evans, David S. "The Relationship between Firm Growth, Size, and Age: Estimates for 100 Manufacturing Industries." *J. Indus. Econ.* 35 (June 1987): 567–81.
- Fama, Eugene F., and French, Kenneth R. "The Cross-Section of Expected Stock Returns." *J. Finance* 47 (June 1992): 427–65.
- . "Common Risk Factors in the Returns on Stocks and Bonds." *J. Financial Econ.* 33 (February 1993): 3–56.
- . "Size and Book-to-Market Factors in Earnings and Returns." *J. Finance* 50 (March 1995): 131–55.
- . "Multifactor Explanations of Asset Pricing Anomalies." *J. Finance* 51 (March 1996): 55–84.
- Fama, Eugene F., and MacBeth, James D. "Risk, Return, and Equilibrium: Empirical Tests." *J.P.E.* 81 (May/June 1973): 607–36.
- Feldman, Mark, and Gilles, Christian. "An Expository Note on Individual Risk without Aggregate Uncertainty." *J. Econ. Theory* 35 (February 1985): 26–32.
- Ferson Wayne E., and Harvey, Campbell R. "Conditional Variables and the Cross Section of Stock Returns." *J. Finance* 54 (August 1999): 1325–60.
- Ferson, Wayne E.; Kandel, Shmuel; and Stambaugh, Robert F. "Tests of Asset Pricing with Time-Varying Expected Risk Premiums and Market Betas." *J. Finance* 42 (June 1987): 201–20.
- Gomes, Joao; Yaron, Amir; and Zhang, Lu. "Asset Pricing Implications for Firms' Financing Constraints." Manuscript. Philadelphia: Univ. Pennsylvania, Wharton School, Finance Dept., 2002.
- Hall, Bronwyn H. "The Relationship between Firm Size and Firm Growth in the U.S. Manufacturing Sector." *J. Indus. Econ.* 35 (June 1987): 583–606.
- Jagannathan, Ravi, and Wang, Zhenyu. "The Conditional CAPM and the Cross-Section of Expected Returns." *J. Finance* 51 (March 1996): 3–53.
- Jermann, Urban J. "Asset Pricing in Production Economies." *J. Monetary Econ.* 41 (April 1998): 257–75.
- Judd, Kenneth L. "The Law of Large Numbers with a Continuum of IID Random Variables." *J. Econ. Theory* 35 (February 1985): 19–25.
- Kaldor, Nicholas. "Capital Accumulation and Economic Growth." In *The Theory*

- of Capital: Proceedings of a Conference Held by the International Economic Association*, edited by Friedrich A. Lutz and Douglas C. Hague. London: Macmillan, 1961.
- Kogan, Leonid. "Asset Prices and Irreversible Real Investment." Manuscript. Cambridge: Massachusetts Inst. Tech., Sloan School Management, 2000.
- . "An Equilibrium Model of Irreversible Investment." *J. Financial Econ.* 62 (November 2001): 201–45.
- Kothari, S. P.; Shanken, Jay; and Sloan, Richard G. "Another Look at the Cross-Section of Expected Stock Returns." *J. Finance* 50 (March 1995): 185–224.
- Lakonishok, Josef; Shleifer, Andrei; and Vishny, Robert W. "Contrarian Investment, Extrapolation, and Risk." *J. Finance* 49 (December 1994): 1541–78.
- Lettau, Martin, and Ludvigson, Sydney. "Resurrecting the (C)CAPM: A Cross-Sectional Test When Risk Premia Are Time-Varying." *J.P.E.* 109 (December 2001): 1238–87.
- Liew, Jimmy, and Vassalou, Maria. "Can Book-to-Market, Size and Momentum Be Risk Factors That Predict Economic Growth?" *J. Financial Econ.* 57 (August 2000): 221–45.
- Lo, Andrew W., and MacKinlay, A. Craig. "Stock Market Prices Do Not Follow Random Walks: Evidence from a Simple Specification Test." *Rev. Financial Studies* 1 (Spring 1988): 41–66.
- MacKinlay, A. Craig. "Multifactor Models Do Not Explain Deviations from the CAPM." *J. Financial Econ.* 38 (May 1995): 3–28.
- Naik, Vasanttilak. "Asset Prices in Dynamic Production Economies with Time-Varying Risk." *Rev. Financial Studies* 7 (Winter 1994): 781–801.
- Pontiff, Jeffrey, and Schall, Lawrence D. "Book-to-Market Ratios as Predictors of Market Returns." *J. Financial Econ.* 49 (August 1998): 141–60.
- Roll, Richard. "A Critique of Asset Pricing Theory's Test: Part I: On Past and Potential Testability of the Theory." *J. Financial Econ.* 4 (March 1977): 129–76.
- Rouwenhorst, K. Geert. "Asset Pricing Implications of Equilibrium Business Cycle Models." In *Frontiers of Business Cycle Research*, edited by Thomas F. Cooley. Princeton, N.J.: Princeton Univ. Press, 1995.
- Stivers, Christopher T. "Firm-Level Return Dispersion and the Future Volatility of Aggregate Stock Market Returns." Manuscript. Athens: Univ. Georgia, Terry Coll. Bus., 2001.
- Vuolteenaho, Tuomo. "What Drives Firm-Level Stock Returns?" *J. Finance* 57 (February 2002): 233–64.
- Zhang, Lu. "The Value Premium." Manuscript. Rochester, N.Y.: Univ. Rochester, Simon Grad. School Bus. Admin., 2002.